which are valid on the same diameter, we arrive at the singular equation studied

$$\frac{1}{\pi i}\int_{-\rho}^{\rho}\delta(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt=F(t_0),\quad -\rho<t<\rho \tag{6.2}$$

Borrowing the expression for $\delta(t)$ from (6.2) (with or without an appropriate solvability condition) and again returning (by means of continuation) to the initial contour L, we find the required density $\mu(t)$.

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Translated by M.D.F.

PMM U.S.S.R., Vol.50, No.6, pp.782-788, 1986
Printed in Great Britain
0021-8928/86 $10.00+0.00

STABILITY OF A GROWING VISCOELASTIC ROD SUBJECTED TO AGEING*

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The stability of a compressed growing rod of viscoelastic material that possesses the property of ageing /1/ is investigated. In conformity with the Chatayev definition of the stability of dynamic systems and the Lyapunov method described in /2/, stability conditions are obtained for a rod growing during a finite time interval, and in finite and semi-infinite time intervals. Some results of a numerical analysis of the behaviour of such a rod are presented in /3/.

1. Variational formulation of the problem of the stability of a growing viscoelastic rod. We consider a rod that grows in both the longitudinal and transverse directions, where its transverse section possesses two axes of symmetry at each time. The law of variation in the rod length as well as the kinematics of its growth in the plane of the transverse section are considered given /1/, whereupon the time of material generation $\tau^*(\rho)$ can be determined in the neighbourhood of a point with coordinates $\rho=\{x, y, z\}$ (the rod length is $l(0)$ at the initial instant). As the time t_1 elapses, the length, the cross-sectional area, and the moment of inertia of the rod remain unchanged and respectively equal to l_0, $F_0(x)$, $J_0(x)$ $(F_0(x)\neq 0, J_0(x)\neq 0)$. A one-parameter conservative compressive load $q(t, x)$ acts on the rod, and causes the normal force $N_0(t, x)=-\beta N_*(t, x)$ therein (β is the load parameter). Axial displacements $u_0(t, x)$, determining the trajectory of the unperturbed motion, appear in the rod subjected to the load in the rectilinear equilibrium position. We assume that when there is no external load the rod axis has a small initial curvature $\alpha w_0(x)$ in the xy plane (α is a small parameter). In this case the rod receives additional displacement αu_1, αw_1 under the effect of the load. We designate the rod motion to which the displacements $u_0+\alpha u_1$, αw_1 correspond to perturbed, and αu_1, αw_1 as perturbations. The rod curvature αw_0 is an external perturbation, with respect to which it is assumed that it is twice differentiable with respect to x, where the first and the second derivatives are square summable in the segment $[0, l(t)]$, where $l(t)$ is the rod length at the running time t.

Definition. The unperturbed rod motion is called stable with respect to the perturbation

**Prikl.Matem.Mekhan.*, 50, 6, 1012-1019, 1986

αw_0 for $0 \leqslant t < \infty$ if for any number $A > 0$ there is a number $\delta = \delta(A) > 0$ such that for any initial curvature αw_0 satisfying the inequality

$$\alpha \sup_x |w_0(x)| < \delta, \quad x \in [0, l(t)]$$

the displacement perturbation αw_1 satisfies the condition

$$\alpha \sup_{t, x} |w_1(t, x)| < A, \quad x \in [0, l(t)]$$

If the rod motion is investigated in a finite time interval $[0, T]$ and a critical value of the deflection $|w|^*$ is given, then in this case it is possible to speak about the critical time t_* by defining it as the time of first reaching the magnitude $|w|^*$ by the deflection $|\alpha w_1|$

$$\alpha \sup_{t, x} |w_1(t, x)| < |w|^*$$

where

$$0 \leqslant t < t^*, \quad \alpha \sup_x |w_1(t_*, x)| = |w|^*, \quad x \in [0, l(t)]$$

The rod is called stable in the interval $[0, T]$ if $t_* > T$.

We take the equation of state for an inhomogeneously ageing viscoelastic material in the uniaxial stress state in the form /4/

$$\sigma(t, \rho) = E(t - \tau^*(\rho)) \varepsilon(t, \rho) - \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho)) \varepsilon(\tau, \rho)\, d\tau \tag{1.1}$$

where σ, ε is the stress and strain in the growing rod, $E(t)$ is the elastic instantaneous strain modulus, and $R(t, \tau)$ is the relaxation kernel of the ageing viscoelastic material.

We will use a modified hypothesis of plane sections /1, 3/ in determing the strain, in conformity with which

$$\begin{gathered}
\varepsilon(t, \rho) = \Delta\varepsilon^\circ(t, x) + \Delta\chi(t, x)\, y \\
\Delta\varepsilon^\circ(t, x) = \varepsilon^\circ(t, x) - \varepsilon^\circ(\tau^*(\rho), x), \quad \Delta\chi(t, x) = \\
\chi(t, x) - \chi(\tau^*(\rho), x) \\
\varepsilon^\circ = \frac{\partial(u_0 + \alpha u_1)}{\partial x} + \frac{1}{2}\left\{\left[\frac{\partial(u_0 + \alpha u_1)}{\partial x}\right]^2 + \alpha^2 \left[\frac{\partial(w_1 + w_0)}{\partial x}\right]^2 - \right. \\
\left. \alpha^2 \left(\frac{\partial w_0}{\partial x}\right)^2\right\}, \quad \chi = -\alpha \frac{\partial^2 w_1}{\partial x^2}
\end{gathered} \tag{1.2}$$

(ε°, χ are the axial strain and curvature of the additional rod curve).

We represent the strain ε° in the form

$$\begin{gathered}
\varepsilon^\circ = \varepsilon_0^\circ + \alpha\varepsilon_1^\circ + \alpha^2\varepsilon_2^\circ \\
\varepsilon_0^\circ = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 \approx \frac{\partial u_0}{\partial x} \\
\varepsilon_1^\circ = \frac{\partial u_1}{\partial x} + \frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} \approx \frac{\partial u_1}{\partial x} \\
\varepsilon_2^\circ = \frac{1}{2}\left(\frac{\partial u_1}{\partial x}\right)^2 + \frac{1}{2}\left\{\left[\frac{\partial(w_1 + w_0)}{\partial x}\right]^2 - \left(\frac{\partial w_0}{\partial x}\right)^2\right\}
\end{gathered}$$

Then

$$\varepsilon = \Delta\varepsilon_0^\circ + \alpha(\Delta\varepsilon_1^\circ + \Delta\chi \cdot y) + \alpha^2 \Delta\varepsilon_2^\circ \tag{1.3}$$

Considering the quasistatic formulation of the problem, we introduce the functional ($V(t)$ is the rod volume) /5/

$$\begin{gathered}
Э = \int_{V(t)} \left(\frac{1}{2} E\varepsilon^2 - \varepsilon R \varepsilon\right) dV - \beta \int_0^{l(t)} q u\, dx \\
E\varepsilon^2 = E(t - \tau^*(\rho))\, \varepsilon^2(t, \rho), \quad \varepsilon R \varepsilon = \varepsilon(t, \rho) \int_{\tau^*(\rho)}^{t} R(t - \tau^*(\rho), \tau - \tau^*(\rho))\, \varepsilon(\tau, \rho)\, d\tau
\end{gathered}$$

Taking (1.3) into account, we expand the functional $Э$ in powers of the parameter α:

$$Э = Э_0 + \alpha Э_1 + \alpha^2 Э_2 + \ldots$$

We will confine ourselves henceforth to the terms of this expansion that are written down.

We vary the functional $Э$ in the displacements u_1, w_1 referred to the running time t (the displacement u_0 corresponding to the unperturbed motion is not varied). That the first variation of the functional $Э$ equals zero is its stationarity condition

$$\delta Э = \alpha \delta Э_1 + \alpha^2 \delta Э_2 = 0 \tag{1.4}$$

$$\delta\vartheta_1 = \int\limits_{V(t)} \delta(\Delta\varepsilon_1^\circ + \Delta\chi\cdot y)(E - \mathbf{R})\Delta\varepsilon_0^\circ\, dV - \int\limits_0^{l(t)} q\delta u_1\, dx$$

$$\delta\vartheta_2 = \int\limits_{V(t)} [\delta\Delta\varepsilon_2^\circ(E - \mathbf{R})\Delta\varepsilon_0^\circ + \delta(\Delta\varepsilon_1^\circ + \Delta\chi\cdot y)(E - \mathbf{R})(\Delta\varepsilon_1^\circ + \Delta\chi\cdot y)\, dV$$

Taking into account that the expressions $(E - \mathbf{R})\,\Delta\varepsilon_0^\circ$, $(E - R)\cdot(\Delta\varepsilon_1^\circ + \Delta\chi\cdot y)$ define the stress σ_0 in unperturbed motion and the stress perturbation σ_1, respectively, we obtain, after integration over the rod cross-sectional area $F(t, x)$,

$$\delta\vartheta_1 = \int\limits_0^{l(t)} \left(N_0 \frac{\partial\delta u_1}{\partial x} - q\delta u_1\right) dx \tag{1.5}$$

$$\delta\vartheta_2 = \int\limits_0^{l(t)} \left[-M_1 \frac{\partial^2\delta w_1}{\partial x^2} + N_0 \frac{\partial(w_1 + w_0)}{\partial x}\frac{\partial\delta w_1}{\partial x}\right] dx +$$

$$\int\limits_0^{l(t)} \left(N_0 \frac{\partial u_1}{\partial x} + N_1\right)\frac{\partial\delta u_1}{dx}\, dx$$

$$M_1(t, x) = \int\limits_{F(t,x)} \sigma_1(t, \rho)\, y\, dF,\ N_1(t, x) = \int\limits_{F(t,x)} \sigma_1(t, \rho)\, dF$$

Because of the rod equilibrium in unperturbed motion, the equality $\delta\vartheta_1 = 0$ is conserved. It then follows from (1.4) that

$$\delta\vartheta_2 = 0 \tag{1.6}$$

Because of the independence of the variations $\delta u_1, \delta w_1$ we obtain two relations from (1.6). From one we find the perturbation of the rod axial displacement u_1 which, as can be shown, is identically zero, while the second relation has the form

$$\int\limits_0^{l(t)} \left[-M_1 \frac{\partial^2\delta w_1}{\partial x^2} + N_0 \frac{\partial(w_1 + w_0)}{\partial x}\frac{\partial\delta w_1}{\partial x}\right] dx = 0 \tag{1.7}$$

Here /1/

$$M_1 = L(t, t, x)\,\chi(t, x) - \int\limits_{\tau_1^*(x)}^{t} \frac{\partial}{\partial\tau} L(t, \tau, x)\,\chi(\tau, x)\, d\tau$$

$$L(t, \tau, x) = \int\limits_{\tau_1^*(x)}^{\tau} L_1(t - \xi, \tau - \xi)\, y^2(\xi)\, dF(\xi)$$

$$\xi = \tau^*(\rho),\ \tau_1^*(x) = \tau_1^*(\rho^\circ),\ \rho^\circ = \{x, 0, 0\}$$

$$L_1(t, \tau) = E(\tau) - \Gamma(t, \tau) > 0,\ \Gamma(t, t) = 0$$

$$\partial L_1(t, \tau)/\partial\tau = R(t, \tau)$$

$$L(t, t, x) = \int\limits_{\tau_1^*(x)}^{t} E(t - \xi)\, y^2(\xi)\, dF(\xi) = EJ_*(t, x)$$

($EJ_*(t, x)$ is the reduced bending stiffness).

2. Stability of a rod in an infinite time interval.

Theorem. If the following conditions are conserved

$$\lim_{t\to\infty} N_*(t, x) = N^*(x) \neq 0, \quad x \in [0, l_0] \tag{2.1}$$

$$\lim_{t\to\infty} E(t) = E_0, \quad L(t, t, x) \neq 0, \quad x \in [0, l(t)]$$

$$\sup_x \left|\frac{\partial}{\partial\tau} L(t, \tau, x)\right| \leqslant R_0(t, \tau)\, J_0(x), \quad \sup_t \int\limits_0^t R_0(t, \tau)\, d\tau = R_0$$

and a function $R_*(t, \tau)$ exists such that as $T \to \infty$

$$\sup_{t \geqslant T} \int\limits_T^t R_*(t, \tau)\, d\tau \to R$$

$$\int\limits_T^t \sup_x \left|\frac{\partial}{\partial\tau} L(t, \tau, x) - R_*(t, \tau)\, J_0(x)\right| d\tau \to 0$$

then the growing rod is stable for $t \in [0, \infty)$ if the load parameter β satisfies the inequality $\beta < \lambda_1 (1 - R/E_0)$. Here λ_1 is the minimal eigenvalue of the homogeneous boundary value problem to which the equation

$$\int_0^{l_0} E_0 J_0(x) w''^2(x)\,dx = \lambda \int_0^{l_0} N^*(x) w'^2(x)\,dx$$

corresponds.

We note that such a boundary value problem is selfadjoint and its eigenvalues are real and positive /6/.

Proof. As the deflection variation δw_1 in (1.7) we take the deflection w_1 itself. Then (1.7) can be represented as follows (we will henceforth omit the subscript 1 and denote the derivative with respect to x by a prime):

$$\int_0^{l(t)} \Big\{L(t,t,x) w''(t,x) - \int_{\tau_1^*(x)}^{t} \frac{\partial}{\partial\tau} L(t,\tau,x) w''(\tau,x)\,d\tau\Big] w''(t,x) - \beta N_*(t,x)[w(t,x) + w_0(x)]' w'(t,x)\Big\}\,dx = 0 \tag{2.2}$$

We introduce the notation

$$\int_0^{l(t)} E_0 J_0(x) w''^2(t,x)\,dx = \| W''(t)\|^2,\ \Omega(t) = \sup_{\tau\in[0,t]} \| W''(\tau)\| \tag{2.3}$$
$$\int_0^{l(t)} E_0 J_0(x) w_0''^2(x)\,dx = \| W_0''(t)\|^2,\ \| W_0''\| = \sup_{\tau\in[0,t_1]} \| W_0''(\tau)\|$$

We represent (2.2) for the time $t > t_1$ as follows

$$\int_0^{l_0} [E_0 J_0(x) w''^2(t,x) - \beta N^*(x) w'^2(t,x)]\,dx = \int_0^{l} \beta N^*(x) w_0'(x) w'(t,x)\,dx + I_1 + I_2 + I_3 \tag{2.4}$$

Here

$$I_1 = \int_0^{l_0} \int_{\tau_1^*(x)}^{t} \frac{\partial}{\partial\tau} L(t,\tau,x) w''(\tau,x)\,d\tau w''(t,x)\,dx$$
$$I_2 = \int_0^{l_0} \beta [N_*(t,x) - N^*(x)][w'(t,x) + w_0'(x)] w'(t,x)\,dx$$
$$I_3 = \int_0^{l_0} [E_0 J_0(x) - L(t,t,x)] w''^2(t,x)\,dx$$

In conformity with conditions (2.1), a $T = T(A)$ exists for any number $A > 0$ such that for $t > T$

$$|E_0 J_0(x) - L(t,t,x)| = \Big| E_0 J_0(x) - \int_{\tau_1^*(x)}^{t} E(t-\xi) y^2(\xi)\,dF(\xi)\Big| < A E_0 J_0(x),\ |N^*(x) - N_*(t,x)| < AN^*(x)$$

Predetermining $w''(\tau,x)$ for $0 \leqslant \tau < \tau_1^*$: $w''(\tau,x) \equiv 0$, we obtain the estimate

$$I_1 \leqslant \int_0^{l_0}\int_0^{T} R_0(t,\tau) J_0(x) w''(\tau,x)\,d\tau w''(t,x)\,dx + (A + R/E_0)\| W''(t)\|\Omega(t) \leqslant (A + R/E_0)\| W''(t)\|\Omega(t) + \| W''(t)\|\Omega(T) R_0/E_0,\quad I_3 \leqslant A\| W''(t)\|^2,$$
$$I_2 \leqslant A\beta\lambda_1^{-1}\| W''(t)\|(\| W''(t)\| + \| W_0''\|)$$

Taking into account the relationship /6/

$$\int_0^{l_0} E_0 J_0(x) w''^2(x)\,dx \geqslant \lambda_1 \int_0^{l_0} N^*(x) w'^2(x)\,dx \tag{2.5}$$

we have in place of (2.4)

$$\left[1-\frac{\beta}{\lambda_1}-A\left(1+\frac{\beta}{\lambda_1}\right)\right]\|W''(t)\|-\left(A+\frac{R}{E_0}\right)\Omega(t)\leqslant \tag{2.6}$$
$$\frac{\beta}{\lambda_1}(1+A)\|W_0''\|+\frac{R_0}{E_0}\Omega(T)$$

It can be shown that the quantity $\Omega(T)$ is bounded.

Let us consider the time interval $[0, t_1]$. On the basis of the theorem of the mean, we can write

$$\int_0^{l(t)} L(t, t, x)\, w''^2(t, x)\, dx = H_1(t, \eta_1)\|W''(t)\|^2$$
$$\int_0^{l(t)} L(t, t, x)\, w_0''^2(x)\, dx = H_0(t, \eta_0)\|W_0''(t)\|^2$$

where

$$H_i(t, \eta_i) = \frac{L(t, t, x)}{E_0 J_0(x)}\Big|_{x=\eta_i}, \quad \eta_i \in [0, l(t)], \quad i = 0,1$$

Taking the relationship (2.5) into account it follows from (2.2) that

$$\|W''(t)\| \leqslant C_1\|W_0''\| + C_2\int_0^t R_0^*(\tau)\|W''(\tau)\|\,d\tau \tag{2.7}$$
$$R_0^*(\tau) = \sup_{t\in[0,t_1]}\frac{R_0(t,\tau)}{E_0}$$
$$C_1 = \sup_t \frac{\beta}{\lambda_1(t)}\sqrt{\frac{H_0(t,\eta_0)}{H_1(t,\eta_0)}}\left(1-\frac{\beta}{\lambda_1(t)}\right)^{-1},$$
$$C_2 = \sup_t \frac{1}{H_1(t,\eta_1)}\left(1-\frac{\beta}{\lambda_1(t)}\right)^{-1}$$

It is assumed here that $\beta < \lambda_1(t)$ for any $t \in [0, t_1]$.

If the function $R_0(t, \tau)$ has a weak singularity for $t = \tau$, then it is first necessary to go over to an iteration series in the inequality (2.7) /7/, which is regular, starting with a certain number n.

Applying the Gronwall-Bellman lemma, we find

$$\|W''(t)\| \leqslant C_1\|W_0''\|\exp\left[C_2\int_0^t R_0^*(\tau)\,d\tau\right], \quad t\in[0, t_1]$$

The boundedness of the function $\Omega(t)$ can be proved analogously for $t_1 < t \leqslant T$.

Therefore $\|W''(t)\| \leqslant \|W_0''\|\,\Phi(t)$ where $\Phi(t)$ is a certain bounded function. Consequently, we have from relationship (2.6)

$$\left[1-\frac{\beta}{\lambda_1}-\frac{R}{E_0}-A\left(2+\frac{\beta}{\lambda_1}\right)\right]\Omega(t)\leqslant\left[\frac{\beta}{\lambda_1}(1+A)+\Phi(T)\right]\|W_0''\| \tag{2.8}$$

On the basis of the theorem of the mean we can write

$$\|W''(t)\| = E_0 J_0(x_0(t))\;\|w''(t)\|^2 \tag{2.9}$$
$$\int_0^{l_0} N^*(x)\, w'^2(t, x)\, dx = N^*(x_1, t)\|w'^2(t)\|$$
$$\|w'(t)\|^2 = \int_0^{l_0} w'^2(t, x)\, dx, \quad \|w''(t)\|^2 = \int_0^{l_0} w''^2(t, x)\, dx$$

Taking account of inequality (2.5) we obtain

$$N^*(x_1(t))\|w'(t)\|^2 \leqslant E_0 J_0(x_0(t))\|w''(t)\|^2/\lambda_1 \tag{2.10}$$

It follows from (2.8)-(2.10), that for

$$\beta < (1 - R/E_0)\lambda_1 \tag{2.11}$$

the function $w(t, x)$ has first and second derivatives square summable in the segment $[0, l_0]$.

Setting the origin at the endpoint where the rod deflection w is zero, we write

$$|w(t, x)| = \left|\int_0^x w'(t, x)\,dx\right| \leqslant \int_0^x |w'(t, x)|\,dx \leqslant \sqrt{l_0}\,\|w'(t)\| \leqslant \sqrt{l_0}\,\sup_t\|w'(t)\|$$

The theorem is proved.

Remark. 1°. If the growing rod is reinforced where the reinforcement material is subject to Hooke's law $\sigma_a = E_a e_a$, then condition (2.11) remains valid even in the case when λ_1 is understood to be the minimal eigenvalue of the homogeneous boundary value problem that corresponds to equality

$$\int_0^{l_0} [E_0 J_0(x) + E_a J_a(x)]\, w''^2(x)\, dx = \lambda \int_0^{l_0} N^*(x)\, w'^2(x)\, dx$$

Here $J_a(x)$ is the reinforcement moment of inertia in the rod transverse section.

2°. If the external load is multiparametric and such that

$$N_*(x) = \sum_{i=1}^{n} \beta_i N_i^*(x)$$

then the condition of rod stability is written as follows

$$\sum_{i=1}^{n} \frac{\beta_i}{\lambda_i} < 1 - \frac{R}{E_0}$$

where λ_i is the minimal eigenvalue of the homogeneous boundary value problem that corresponds to the equality

$$\int_0^{l_0} E_0 J_0(x)\, w''^2(x)\, dx = \lambda \int_0^{l_0} N_i^*(x)\, w'^2(x)\, dx$$

3°. If the material characteristics are invariant relative to the time origin, i.e., $L_1(t, \tau) = E_0 - \Gamma(t - \tau)$, then for $\tau > t_1$

$$\frac{\partial}{\partial \tau} L(t, \tau, x) = R(t - \tau) J_0(x), \quad R = \int_0^{\infty} R(\theta)\, d\theta, \quad R(t - \tau) = \frac{\partial \Gamma(t - \tau)}{\partial (t - \tau)}$$

In this case the critical value of the load parameter is determined for the growing rod exactly as for an elastic rod with a extended elastic modulus $E_* = E_0 - R$.

3. Stability of a growing rod in a finite time interval. The stability of a growing viscoelastic rod in a finite time interval $[0, T]$ can be investigated by using the relationship (2.7) from which estimates of the critical time t_* or the critical values of the other parameters, particularly the values of the velocities characterizing the rod growth, etc., are obtained for given estimates of the initial curvature w_0 and the ultimate deflection w^*.

As an illustration, we will examine a rod for which one end is rigidly clamped (for $x = 0$) and the other is free (for $x = l(t)$). The initial curvature of the rod axis is described by a parabola $w_0(x) = ax^2$. The rod has a constant cross-section and is under its own weight. Growth of the rod occurs in the axial direction at an exponential velocity $v(t) = v_0 e^{-\alpha t}$.

We note that the normal force at the rod free end is free, however, this is not reflected in the final results.

The material elastic modulus is constant, equal to E_0, while the relaxation kernel has the form /1/

$$R(t, \tau) = -\frac{\partial}{\partial \tau} \{\omega(\tau)[1 - e^{-\gamma(t-\tau)}]\}, \quad \omega(\tau) = C_0 + A_0 e^{-b\tau}$$

Under the assumptions made, the following equalities hold

$$H_i(t, \eta_i) \equiv 1, \ \| W''(t) \|^2 = E_0 J_0 \| w''(t) \|^2$$
$$\| W_0''(t) \|^2 = E_0 J_0 \| w_0''(t) \|^2 = E_0 J_0 (2a \sqrt{l(t)})^2$$

Relationship (2.7) becomes

$$\left(1 - \frac{q}{\lambda_1(t)}\right) \| w''(t) \| \leqslant \frac{q}{\lambda_1(t)} 2a \sqrt{l(t)} + \int_0^t \frac{R_0(t, \tau)}{E_0} \| w''(\tau) \|\, d\tau \tag{3.1}$$
$$\lambda_1(t) \approx 7.84 E_0 J_0 / l^3(t)$$

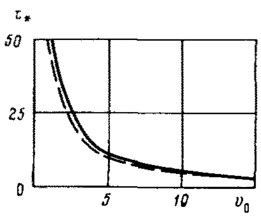
An estimate of the magnitude of the rod deflection can be obtained in the norm $\| w''(t) \|$

$$|w'(t, x)| = \left| \int_0^x w''(t, x)\, dx \right| \leqslant \int_0^x |w''(t, x)|\, dx \leqslant l^{1/2}(t) \frac{\Omega(t)}{E_0 J_0}$$
$$|w(t, x)| = \left| \int_0^x w'(t, x)\, dx \right| \leqslant l^{3/2}(t) \frac{\Omega(t)}{E_0 J_0} \tag{3.2}$$

Using the relationships (3.1) and (3.2), an estimate can be found for the value of the critical time t_*. Here

$$R_0(t, \tau) = bA_0 + (\gamma C_0 + \gamma A_0 - bA_0)\, e^{-\gamma(t-\tau)}$$

The results of solving the problem in the form of the dependence of t_* on the parameter v_0, found for values of the constants $l(0) = 0$, $C_0/E_0 = 0.075$, $A_0/E_0 = 0.75$, $\gamma = 0.02 \cdot 1$/day, $b = 0.005 \cdot 1$/day, $l_0 = v_0/\alpha = 50$ m, $q = 0.255\lambda_1$, $a = 4 \cdot 10^{-5}\,\mathrm{m}^{-1}$, are shown in the figure as a continuous curve that corresponds to the value of the ultimate deflection w^*, equal to 0.01m (the time t_* is measured in days, and v_0 in m/day). The graph shows the substantial influence of the rod growth rate on the magnitude of the critical time.



The dependece of w^* on t_* can be obtained on a more lucid form for which we write inequality (3.1) as follows

$$\| w''(t) \| \leqslant \Phi(t) + \varphi(t) \int_0^t R_0^*(\tau) \| w''(\tau) \| \, d\tau$$

$$\varphi(t) = [1 - q\lambda_1^{-1}(1 - e^{-\alpha t})^3]^{-1}, \quad \Phi(t) = 2\, a l_0^{1/2} q \lambda_1^{-1} (1 - e^{-\alpha t})^{3.5} \varphi(t),$$
$$R_0^*(\tau) = \gamma (C_0 + A_0) / E_0$$

Taking account of the monotonic change in the functions $\varphi(t)$ and $\Phi(t)$ as the time t increases, we have the following transcendental equation for the estimate of t_*:

$$w^* = l_0^{3/2} \Phi(t_*)(1 - e^{-\alpha t_*})^{1/2} \exp[t_* \varphi(t_*) \gamma (C_0 + A_0)/E_0] \tag{3.3}$$

Expressing α in terms of v_0 for a fixed value of w^* we hence obtain a dependence between t_* and the parameter v_0 which is shown dashed in the figure. Comparison of the graphs confirms the closeness of the results that the relationships (3.1), (3.2) and (3.3) yield.

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Translated by M.D.F.